A new formula involving the Euler-Mascheroni constant, and generalization

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Abstract

I use two methods to evaluate an ordinary first-order non-homogeneous differential equation, and evaluate a limit at a removable singularity to reveal a new series representation for the difference $\gamma - \mathrm{Ei}(1)$ of the Euler-Mascheroni constant and the exponential integral function evaluated at 1.

Note that the formulas presented in this paper look very similar to some formulas (ffff) found on Wolfram Mathworld, Wolfram Alpha and Wikipedia. However, as the reader may verify, none of the formulas presented in this paper seem to be trivial consequences of the cited formulas, or vice versa.

The techniques used in proving the formulas in this paper are very elementary. So at times I will omit some details which are the sorts of things that are covered in an introductory analysis class.

1 Conventions and notations

An "empty sum" of the form $\sum_{k=0}^{-1} a_k$ is equal to zero.

2 Results

The main results of this paper may be summed up as follows:

A new formula:

$$\sum_{n=1}^{\infty} \frac{1}{nn!} = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \left(\frac{k!}{n!} - \frac{1}{n^2} \right) \tag{1}$$

An identity (which holds for positive integers m) generalizing formula ffffff:

$$\sum_{n=1}^{\infty} \frac{1}{nn!} = \sum_{n=1}^{\infty} \left(-\frac{(m-1)!(n-1)!}{(n+m-1)!} + \left((m-1)! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{k_1-1} \cdots \sum_{k_m=0}^{k_{m-1}-1} \frac{k_m!}{n!} \right) - \sum_{l=1}^{m-1} \frac{(-1)^l}{n(n+l)!(m-l-1)!} \right) \tag{2}$$

And a further generalization which provides an analogous identity to ffff for each non-negative integer l

$$\sum_{n=1}^{\infty} \frac{1}{n(n+r)!} = \sum_{n=1}^{\infty} \left(-\frac{(m-r-1)!(n-1)!}{(n+m-1)!} + \left((m-r-1)! \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{k_1-1} \cdots \sum_{k_m=0}^{k_{m-1}-1} \frac{k_m!}{n!} \right) \right) - \sum_{n=1}^{\infty} \sum_{l=1, l\neq r}^{m-1} \frac{(-1)^l}{n(n+l)!(m-l-1)!} \tag{3}$$

The relationship between fffff, the Euler-Mascheroni constant and the exponential integral is by the already-known formula:

$$\gamma - \mathrm{Ei}(1) = \sum_{n=1}^{\infty} \frac{1}{nn!}. \tag{4}$$

where γ is the Euler-Mascheroni constant, defined

$$\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \int_1^{N+1} \frac{1}{t}\, \mathrm{d}t \right) \tag{5}$$

and where $\mathrm{Ei}(1)$ is defined

$$\mathrm{Ei}(1) = \int_{-\infty}^{1} \frac{e^t}{t}\, \mathrm{d}t \tag{6}$$

Thus fffff gives us a new series representation of $\gamma - \mathrm{Ei}(1)$. The reader should convince themself that one side of ffffff is not a trivial manipulation of the terms of the other side.

For the sake of readability, we will first prove ffffff and prove its generalization separately. In the proof of ffffffff we will also prove the base case of an inductive argument used in fffffff.

3 Proof of first formula

We begin by considering the first-order ordinary non-homogeneous differential equation

$$y' - y = \frac{e}{1-z} \quad \text{and} \quad y(0) = 0 \tag{7}$$

where $y = y(z)$ is defined in a neighborhood U of 0 in $\mathbb{C}$, and e is the base of the natural logarithm.

By the fundamental existence and uniqueness theorem of differential equations, a unique solution exists, and since the derivative of y is assumed to exist in U, y is analytic on U. Since y is analytic, first we solve ffffff by computing the coefficients of the Taylor series of y. Second, we will solve fffff by integrating factor.

3.1 Taylor series solution

Since y is infinitely differentiable, and its derivatives exist at 0,

$$y^{(n+1)}(0) - y^{(n)}(0) = en! \quad \text{and} \quad y(0) = 0 \tag{8}$$

This is a first-order linear constant-coefficient recurrence relation with initial condition. Its solution is easily found by induction on n to be

$$y^{(n)}(0) = e \sum_{k=0}^{n-1} k! \quad \text{for } n > 1; \;\; y(0) = 0 \tag{9}$$

Let's adopt the convention that $\sum_{k=0}^{-1} k! = 0$ for convenience. Then

$$y(z) = e \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \frac{k!}{n!} z^n \tag{10}$$

on U.

3.2 Solution by integrating factor

Now we solve by using the integrating factor $\mu = e^{-z}$

$$y(z) = e^z \int_0^z \frac{e^{1-t}}{1-t}\, \mathrm{d}t \tag{11}$$

$$= e^x \int_0^x \sum_{n=0}^{\infty} \frac{(1-t)^{n-1}}{n!}\, \mathrm{d}t \tag{12}$$

and by uniform continuity

$$= e^x \sum_{n=0}^{\infty} \int_0^x \frac{(1-t)^{n-1}}{n!}\, \mathrm{d}t \tag{13}$$

$$= e^x \left[-\sum_{n=1}^{\infty} \frac{(1-x)^n}{nn!} + \sum_{n=1}^{\infty} \frac{1}{nn!} - ln|1-x| \right] \tag{14}$$

3.3 Combining the solutions

Now using the two expressions for y, we get

$$0 = y(x) - y(x) = -e \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{k!}{n!} x^n + e^x \left[- \sum_{n=1}^{\infty} \frac{(1-x)^n}{nn!} + \sum_{n=1}^{\infty} \frac{1}{nn!} - ln|1-x| \right] \tag{15}$$

Note that the function $f(x) = \sum_{n=1}^{\infty} \frac{(1-x)^n}{nn!}$ is continuous at 1 and that f(1)=0.
Now we re-express the logarithm as a series

$$ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \tag{16}$$

Now rearranging fffffff and using sum and limit properties, we get

$$\sum_{n=1}^{\infty} \frac{1}{nn!} = \sum_{n=1}^{\infty} \left(-\frac{1}{n} + \sum_{k=0}^{n-1} \frac{k!}{n!} \right) \tag{17}$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \left(-\frac{1}{n^2} + \frac{k!}{n!} \right) QED \tag{18}$$

4 Proof of generalized formula

The proof of the generalized formula is very similar to the $m = 1$ case, except that induction will be required in a few spots. We consider a generalization of differential equation ffffffff:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - 1 \right)^{(m)} y_m = \frac{e}{1-z}, \quad \text{and } y_m(0) = \cdots = y_m^{(m)}(0) = 0 \tag{19}$$

where $\left(\frac{\mathrm{d}}{\mathrm{d}x} - 1 \right)^{(m)}$ is the m-fold differential operator, and $y_m^{(m)}$ is the mth derivative of y_m.

The coefficients to the $y_m^{(i)}$ term on the left-hand-side is $(-1)^i \binom{m}{i}$

4.1 Taylor series solution

Differentiating both sides n times and evaluating at 0, we get

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} y_m^{(n+m-i)}(0) = en! \quad \text{and} \quad y_m(0) = \cdots = y_m^{(m)}(0) = 0 \tag{20}$$

This is an mth order linear constant-coefficient non-homogeneous difference equation with m initial values. The form of the left-hand-side makes it possible to solve (to find an explicit formula for $y_m^{(n)}(0)$) by induction on m.

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Now we find the solution to difference equation ffff by induction on m. For the base case, recall that in section ffff we proved that the difference equation

$$y_1^{(n+1)}(0) - y_1^{(n)}(0) = en! \quad \text{and} \quad y_1(0) = 0 \tag{21}$$

had the unique solution

$$y_1^{(n)}(0) = e\sum_{k=0}^{n-1} k! \tag{22}$$

Now, generalizing our base case, suppose that for a fixed m,

$$y_{m-1}^{(n)}(0) = e\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_{m-1}=0}^{k_{m-2}-1} k_{m-1}! \tag{23}$$

Then noting that $y_{m-1}(z) = y_m'(z) - y_m(z)$

$$y_m^{(n+1)}(0) = y_m^{(n)}(0) + e\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_{m-1}=0}^{k_{m-2}-1} k_{m-1}! \tag{24}$$

It is simple enough to show by inducting on n that

$$y_m^{(n)}(0) = e\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_m=0}^{k_{m-1}-1} k_m! \tag{25}$$

so we'll let this be an exercise for the reader. This completes our induction on m. And so we obtain a Taylor series solution for y_m in a neighborhood of 0:

$$y_m(z) = e\sum_{n=1}^{\infty}\left(\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_m=0}^{k_{m-1}-1}\frac{k_m!}{n!}x^n\right) \tag{26}$$

4.2 Solution by integrating factor

To obtain a solution for y_m by integrating factor, again we generalize the method used in the $m=1$ case by inducting on a parameter $j = 0, 1, ..., m-1$. The base case ($j=0$), by the same reasoning as in section ffffff is:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^{(m-1)} y_m = e^x\left[-\sum_{n=1}^{\infty}\frac{(1-x)^n}{nn!} + \sum_{n=1}^{\infty}\frac{1}{nn!} + \sum_{n=1}^{\infty}\frac{x^n}{n}\right] \tag{27}$$

Suppose that for some j such that $0 \le j \le m-2$ (will prove it works for all j from 0 to m-1 including m-1),

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}-1\right)^{(m-j-1)} y_m = e^x\left[(-1)^{j+1}\sum_{n=1}^{\infty}\frac{(1-x)^{n+j}}{n(n+j)!}+\sum_{l=0}^{j}\sum_{n=1}^{\infty}\frac{(-1)^l x^{j-l}}{n(n+l)!(j-l)!}+\sum_{n=1}^{\infty}\frac{(n-1)!}{(n+j)!}x^{n+j}\right] \tag{28}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x}\left(\frac{\mathrm{d}}{\mathrm{d}x}-1\right)^{(m-j-2)} y_m\right) = (-1)^{j+1}\sum_{n=1}^{\infty}\frac{(1-x)^{n+j}}{n(n+j)!}+\sum_{l=0}^{j}\sum_{n=1}^{\infty}\frac{(-1)^l x^{j-l}}{n(n+l)!(j-l)!}+\sum_{n=1}^{\infty}\frac{(n-1)!}{(n+j)!}x^{n+j} \tag{29}$$

As in section ffffff,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}-1\right)^{(m-(j+1)-1)} y_m \tag{30}$$

$$= e^x\left[(-1)^{(j+1)+1}\left(\sum_{n=1}^{\infty}\frac{(1-x)^{n+(j+1)}}{n(n+(j+1))!}-\sum_{n=1}^{\infty}\frac{1}{n(n+(j+1))!}\right)+\sum_{l=0}^{j}\sum_{n=1}^{\infty}\frac{(-1)^l x^{(j+1)-l}}{n(n+l)!((j+1)-l)!}+\sum_{n=1}^{\infty}\frac{(n-}{(n+(j} \tag{31}$$

$$= e^x\left[(-1)^{(j+1)+1}\sum_{n=1}^{\infty}\frac{(1-x)^{n+(j+1)}}{n(n+(j+1))!}+\sum_{l=0}^{j+1}\sum_{n=1}^{\infty}\frac{(-1)^l x^{(j+1)-l}}{n(n+l)!((j+1)-l)!}+\sum_{n=1}^{\infty}\frac{(n-1)!}{(n+(j+1))!}x^{n+(j+1)}\right] \tag{32}$$

So the inductive argument is complete, and substituting $j = m-1$ into equation ffffff we get

$$y_m = e^x\left[(-1)^{m}\sum_{n=1}^{\infty}\frac{(1-x)^{n+m-1}}{n(n+m-1)!}+\sum_{l=0}^{m-1}\sum_{n=1}^{\infty}\frac{(-1)^l x^{m-l-1}}{n(n+l)!(m-l-1)!}+\sum_{n=1}^{\infty}\frac{(n-1)!}{(n+m-1)!}x^{n+m-1}\right] \tag{33}$$

4.3 Combining the solutions

As before, we combine the solutions and evaluate the limit as $x \to 1$ from within the convergence zone.

$$0 = y(x) - y(x) \tag{34}$$

$$= e\sum_{n=1}^{\infty}\left(\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_m=0}^{k_{m-1}-1}\frac{k_m!}{n!}x^n\right)-e^x\left[(-1)^{m}\sum_{n=1}^{\infty}\frac{(1-x)^{n+m-1}}{n(n+m-1)!}+\sum_{l=0}^{m-1}\sum_{n=1}^{\infty}\frac{(-1)^l x^{m-l-1}}{n(n+l)!(m-l-1)!}+\sum_{n=1}^{\infty}\frac{}{(n} \tag{35}$$

Evaluating the limit and shifting stuff around

$$\sum_{l=0}^{m-1}\sum_{n=1}^{\infty}\frac{(-1)^l}{n(n+l)!(m-l-1)!}=\sum_{n=1}^{\infty}\left(-\frac{(n-1)!}{(n+m-1)!}+\left(\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_m=0}^{k_{m-1}-1}\frac{k_m!}{n!}\right)\right) \tag{36}$$

For each non-negative integer r, equation ffff gives us the following identity which holds for all positive integers m:

$$\sum_{n=1}^{\infty}\frac{1}{n(n+r)!}=\sum_{n=1}^{\infty}\left(-\frac{(m-r-1)!(n-1)!}{(n+m-1)!}+\left((m-r-1)!\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_m=0}^{k_{m-1}-1}\frac{k_m!}{n!}\right)\right)-\sum_{n=1}^{\infty}\sum_{l=1,l\neq r}^{m-1}\frac{}{n(n+l} \tag{37}$$

for any non-negative integer r and for any positive integer m which. As far as I can see, the various formulas for $\sum\frac{1}{n(n+r)!}$ for a fixed r and varying m are not pairwise related by some trivial symbolic manipulation.

Plugging in $l=0$ gives the following identity:

$$\gamma-\mathrm{Ei}(1)=\sum_{n=1}^{\infty}\left(-\frac{(m-1)!(n-1)!}{(n+m-1)!}+\left((m-1)!\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{k_1-1}\cdots\sum_{k_m=0}^{k_{m-1}-1}\frac{k_m!}{n!}\right)-\sum_{l=1}^{m-1}\frac{(-1)^l}{n(n+l)!(m-l-1)!}\right) \tag{38}$$

for any positive integer m.